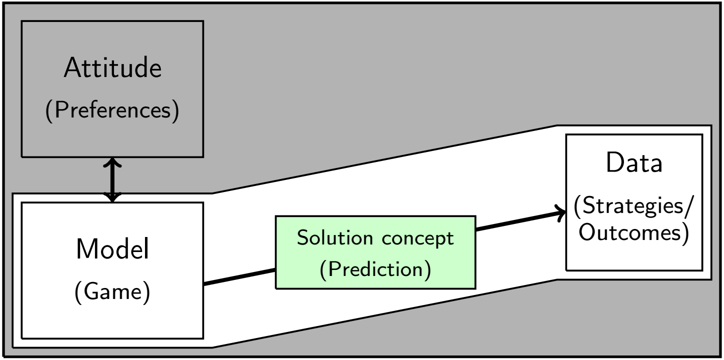
Mechanism design

Theory of Individual and Strategic Decisions

MSc Human Decision Science

Maastricht University

- **Game Theory**: aim is to predict what people do
  1. Start with game description
  2. Make assumptions on strategic reasoning
  3. Predict behavior
- **Mechanism design**: aim is that people willingly do what we want
  1. Start with desired behavior
  2. Make assumptions on strategic reasoning
  3. Find the game that leads to desired behavior



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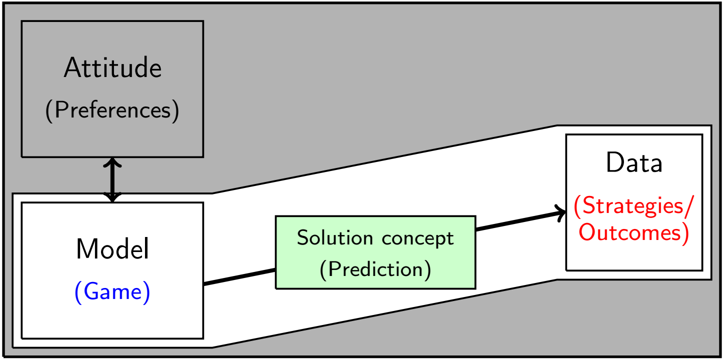

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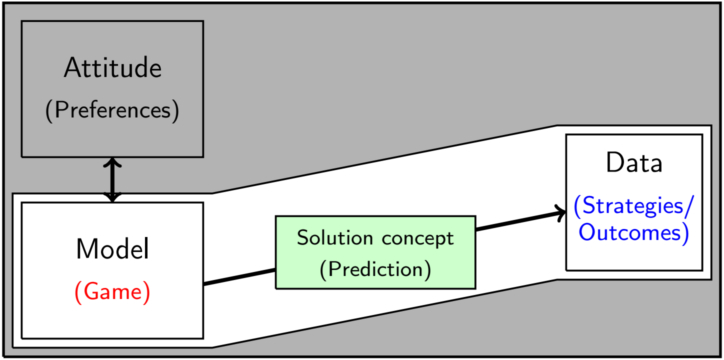


Purpose is usually to solve some allocation problem

Example (Allocating an object)

We want to sell a painting to the buyer who most want it.

Example (Choosing a president)

We want to decide which candidate to be elected so that society is best represented.

Example (Public project)

We want to decide which candidate to be elected so that society is best represented.

Example (Doctor allocation)

We want to decide which doctor will be allocated to each hospital for a residency so that no doctors-hospital pair wants to make a side deal.

- **Key difficulty:** we do not know people's unobservable types
  - In the painting example, their willingness to pay.
  - In the election example, their political preferences
  - In the project example, their personal valuation
  - In the doctor example, the individual preferences

  If we knew all this, the problem would be easy
- The objective will be to to make them reveal their type
- In the lecture/book, we use project example.

- **Direct mechanism (with transfers):** a game where
  1. Players: **citizens** $N = \{1, \ldots, n\}$
  2. Strategy: player i's **report** $x^i \in \mathbb{R}$ of private valuation $v^i \in \mathbb{R}$
  3. Outcome: **decision** $\delta \in \{0, 1\}$ plus **monetary transfer** $\tau^i \in \mathbb{R}$
- **Assumption:** preferences are represented by utility

$$u^i(x^1, \ldots, x^n) = \underbrace{\delta(x^1, \ldots, x^n)}_{\text{decision}} \cdot v^i + \underbrace{\tau^i(x^1, \ldots, x^n)}_{\text{transfer for } i}$$

- Outcomes depend only on what is observable (i.e., the reports)
- **Objective**: we want the project if it truly beneficial for society
  1. All citizens report truthfully:
  $$x^i = v^i$$
  2. Implement if and only if total value positive:
  $$v^1 + \cdots + v^n > 0$$
- **Our problem**: Design a mechanism (i.e., choose δ and τ^i) that meets these objectives

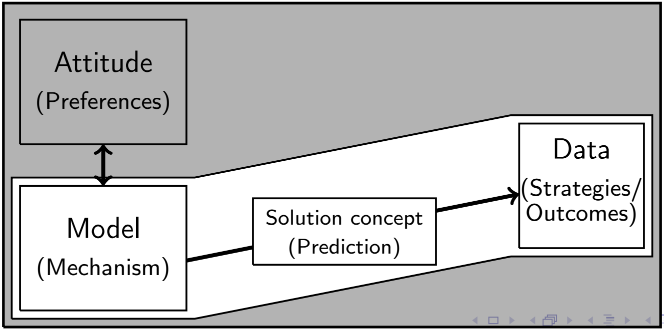
- First objective (all citizens report truthfully): $x^i = v^i$
- Choose mechanism such that reports are truthful
- We use solution concept that makes strong prediction

Definition

Truthful reporting is weakly dominant if, for all x^i and all x^{-i}

$$\underbrace{\delta(v^i, x^{-i}) \cdot v^i + \tau^i(v^i, x^{-i})}_{\text{utility from true report}} \geq \underbrace{\delta(x^i, x^{-i}) \cdot v^i + \tau^i(x^i, x^{-i})}_{\text{utility from other report}}$$

Strategy proof mechanism: truthful reporting is weakly dominant



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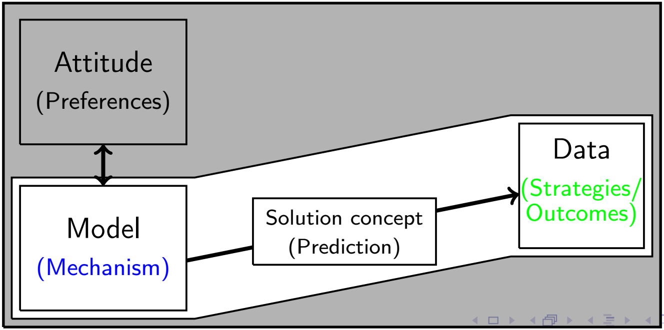

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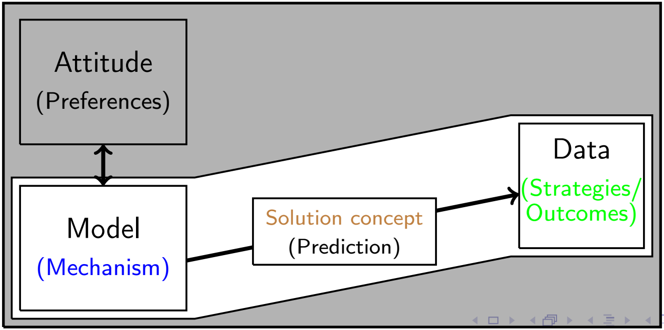

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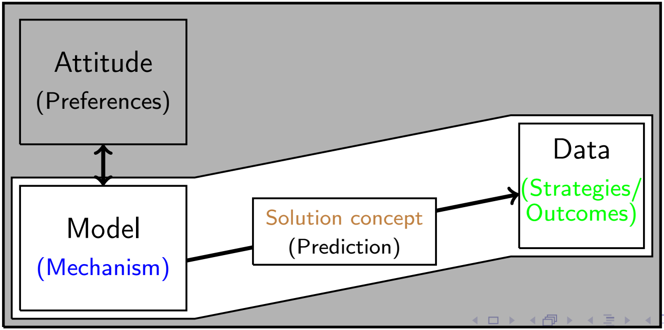

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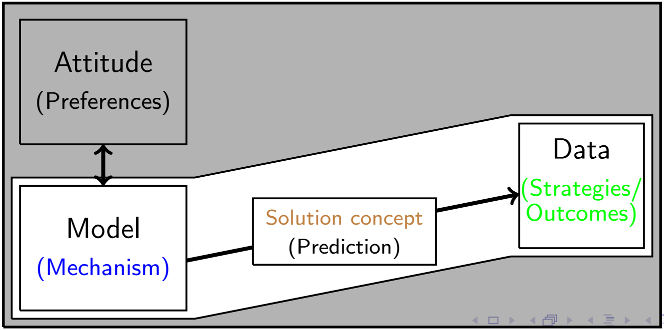

- Second objective (implement good projects): $v^1 + \cdots + v^n > 0$

Proposition

In a strategy proof mechanism, second objective is met if

$$\boxed{\delta(x^1, \ldots, x^n) = 1 \;\Leftrightarrow\; x^1 + \cdots + x^n > 0}$$

- Let's see if some mechanisms reach our objectives.
- Any ideas of mechanisms that are obvious candidates?

Example (Majority voting)

- A positive report counts for one vote in favor, and a negative report counts for one vote against (regardless of the size of x^i)
- The project is implemented if more than half votes in favor

$$\delta(x^1, \ldots, x^n) = 1 \;\Leftrightarrow\; \Big|\underbrace{\{i \in N : x^i > 0\}}_{\text{votes in favor}}\Big| > n/2$$

- There are no transfers: $\tau^i(x^1, \ldots, x^n) = 0$

Proposition

Majority voting is strategy proof, but does not necessarily implement good projects.

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Majority voting is strategy proof, but does not necessarily implement good projects. Explain. Why? Give an example.

Example

Project: Tear Carol's house down in order to plant a beautiful tree in our street. Three citizens with $v^a = 1$, $v^b = 1$, $v^c = -1000$

Example (Summing up reports)

- The project is implemented if sum of reports is positive

$$\delta(x^1, \dots, x^n) = 1 \iff x^1 + \cdots + x^n > 0$$

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Summing up reports is not strategy proof. Why? Give an example.

Example

Project: Move the public parking spot in front of Carol's house 20 meters down the road in order to plant a beautiful tree. Three citizens with $v^a = 1$, $v^b = 1$, $v^c = -1$. Carol will exaggerate with her valuation (e.g., she will report $x^c = -10$) in order for the project not to be implemented.

Example (Vickrey-Clarke-Groves (VCG))

- The project is implemented if sum of reports is positive

$$\delta(x^1, \dots, x^n) = 1 \;\Leftrightarrow\; x^1 + \cdots + x^n > 0$$

- The transfers look as follows:

$$\tau^i(x^i, x^{-i}) = \begin{cases} \bar{x}^{-i} & \text{if } \bar{x}^{-i} \leq 0 \text{ and } x^i + \bar{x}^{-i} > 0 \\ -\bar{x}^{-i} & \text{if } \bar{x}^{-i} > 0 \text{ and } x^i + \bar{x}^{-i} \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

  where $\bar{x}^{-i} = \sum_{j \neq i} x^j$ is the sum of everyone else's report.

  1. If i is pivotal in favor of the project, he pays $-\bar{x}^{-i} \leq 0$
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- **Key idea:** It is costly to be the deciding factor.

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	Ann	Bob	Carol	decision
valuation	1	2	-2	
report				
utility				

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valuation	1	2	-2	
report	1	4	-2	
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utility				

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utility	1	$2 - 1$	-2	

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utility	1 − 1	2 − 2	−2	

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valuation	1	2	-2	
report	2	3	-8	
utility				

$$\tau^i(x^i, x^{-i}) = \begin{cases} \bar{x}^{-i} & \text{if } \bar{x}^{-i} \leq 0 \text{ and } x^i + \bar{x}^{-i} > 0 \\ -\bar{x}^{-i} & \text{if } \bar{x}^{-i} > 0 \text{ and } x^i + \bar{x}^{-i} \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

	Ann	Bob	Carol	decision
valuation	1	2	−2	
report	2	3	−8	✗
utility				

$$\tau^i(x^i, x^{-i}) = \begin{cases} \bar{x}^{-i} & \text{if } \bar{x}^{-i} \leq 0 \text{ and } x^i + \bar{x}^{-i} > 0 \\ -\bar{x}^{-i} & \text{if } \bar{x}^{-i} > 0 \text{ and } x^i + \bar{x}^{-i} \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

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	Ann	Bob	Carol	decision
valuation	1	2	-2	
report	2	3	-8	✗
utility	0	0	$0 - 5$	

Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	✓
report	1	x^b	x^c	

$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	
report	x^a	x^b	x^c	

$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

1. $x^a > -x^b - x^c$:

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$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

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Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	✓
report	x^a	x^b	x^c	

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1. $x^a > -x^b - x^c$: decision still positive, Ann still pivotal.
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Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	✗
report	x^a	x^b	x^c	

$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

1. $x^a > -x^b - x^c$: decision still positive, Ann still pivotal.
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2. $x^a \leq -x^b - x^c$: decision negative,

Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	✗
report	x^a	x^b	x^c	

$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

1. $x^a > -x^b - x^c$: decision still positive, Ann still pivotal.

$$u^a(x^a, x^b, x^c) = 1 + x^b + x^c = u^a(1, x^b, x^c)$$

2. $x^a \leq -x^b - x^c$: decision negative, Ann not pivotal.

Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	2	-2	✗
report	x^a	x^b	x^c	

$$u^a(1, x^b, x^c) = 1 + x^b + x^c$$

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Theorem

The VCG mechanism is strategy proof.

	Ann	Bob	Carol	decision
valuation	1	[illegible]	[illegible]	[illegible]
report	x	x	x	

Reporting 1 is weakly dominant

$$\boxed{u^a(1, x^b, x^c) = 1 + x^b + x^c}$$

1. $x^a > -x^b - x^c$: decision still positive, Ann still pivotal.
$$u^a(x^a, x^b, x^c) = 1 + x^b + x^c = u^a(1, x^b, x^c)$$
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$$u^a(x^a, x^b, x^c) = 0 < 1 + x^b + x^c = u^a(1, x^b, x^c)$$

Problems with VCG mechanism:

1. Some people pay even if the project is not implemented.
2. It is difficult to explain the mechanism to people.
3. Payments are not distributed back to individuals.

- **Auction:** a game where
  1. Players: **bidders** $N = \{1, \dots, n\}$
  2. Strategy: player i's **bid/report** $x^i \geq 0$ of valuation $v^i \geq 0$
  3. Outcome: **allocation** $w^i \in \{0,1\}$ plus **payment** $\tau^i \geq 0$
- **Assumption:** preferences are represented by utility

$$u^i(x^1, \dots, x^n) = \underbrace{w^i(x^1, \dots, x^n)}_{\text{winner}} \cdot v^i - \underbrace{\tau^i(x^1, \dots, x^n)}_{\text{payment}}$$

- **Objective**:
  1. All bidders report truthfully:
     $$x^i = v^i$$
  2. Bidder with the highest valuation wins the object:
     $$w^i = 1 \;\Rightarrow\; v^i \geq v^j \text{ for all } j \in N$$
- **Our problem**: Design a mechanism (i.e., choose δ and τ^i) that meets these objectives

Example (First-price auction)

- Highest bidder wins the object (ties broken randomly)
- Winner pays the highest bid

Proposition

First-price auction is not strategy proof.

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First-price auction is not strategy proof. Why? Give an example.

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- Highest bidder wins the object (ties broken randomly)
- Winner pays the highest bid

Proposition

First-price auction is not strategy proof. Why? Give an example.

Example

Two bidders with $v^a = 30$, $v^b = 20$. If $x^b = 20$ then we have

$$u^a(25, x^b) > u^a(v^a, x^b).$$

Example (Second-price auction)

- Highest bidder wins the object (ties broken randomly)
- Winner pays the bid of the second highest bidder (if tied on the top, pays his own bid)

Proposition

Second-price auction is strategy proof.

- **Individuals:** from two groups, X and Y.
  - For simplicity, assume $|X| = |Y|$
- **Preferences** about individuals in other group:
  - Each individual in X has strict preferences over Y (x's type)
  - Each individual in Y has strict preferences over X (y's type)
- **Example:** Suppose that $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 c \succ^2 b$ Preferences of b : $1 \succ^b 3 \succ^b 2$

Preferences of 3 : $b \succ^3 c \succ^3 a$ Preferences of c : $3 \succ^c 2 \succ^c 1$

Example (House allocation)

There are people who want to rent (X) and houses available (Y).

Example (Doctors)

There are doctors who wants to do their residency (X) and hospitals looking for doctors (Y).

Example (Marriage market)

There are men (X) and women (Y) who want to find a spouse.

- **Matching:** one-to-one function $\mu : X \to Y$
  - A match is a pair $(x, \mu(x))$.
- **Stable matching:** there are no incentives for side agreements

$$y \succ^x \mu(x) \text{ and } x \succ^y \mu^{-1}(y)$$

In this case, we say that (x, y) is a blocking pair.

Preferences of 1 : $a \succ^1 b \succ^1 c$

Preferences of 2 : $a \succ^2 c \succ^2 b$

Preferences of 3 : $b \succ^3 c \succ^3 a$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $1 \succ^b 3 \succ^b 2$

Preferences of c : $3 \succ^c 2 \succ^c 1$

Matchings			**Blocking pair**	**Stable**
(a,1)	(b,2)	(c,3)	$(a, 2)$. Why?	✗
(a,1)	(b,3)	(c,2)	$(a, 2)$. Why?	✗
(a,2)	(b,1)	(c,3)	None. Why?	✓
(a,2)	(b,3)	(c,1)	$(b, 1)$. Why?	✗
(a,3)	(b,1)	(c,2)	$(a, 2)$. Why?	✗
(a,3)	(b,2)	(c,1)	$(a, 2)$. Why?	✗

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Preferences of b : $1 \succ^b 3 \succ^b 2$

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Matchings			Blocking pair	Stable
(a,1)	(b,2)	(c,3)	$(a, 2)$. Why?	✗
(a,1)	(b,3)	(c,2)	$(a, 2)$. Why?	✗
(a,2)	(b,1)	(c,3)	None. Why?	✓
(a,2)	(b,3)	(c,1)	$(b, 1)$. Why?	✗
(a,3)	(b,1)	(c,2)	$(a, 2)$. Why?	✗
(a,3)	(b,2)	(c,1)	$(a, 2)$. Why?	✗

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Preferences of $1 : a \succ^1 b \succ^1 c$

Preferences of $2 : a \succ^2 c \succ^2 b$

Preferences of $3 : b \succ^3 c \succ^3 a$

Preferences of $a : 2 \succ^a 1 \succ^a 3$

Preferences of $b : 1 \succ^b 3 \succ^b 2$

Preferences of $c : 3 \succ^c 2 \succ^c 1$

Matchings			Blocking pair	Stable
(a,1)	(b,2)	(c,3)	$(a, 2)$. Why?	✗
(a,1)	(b,3)	(c,2)	$(a, 2)$. Why?	✗
(a,2)	(b,1)	(c,3)	None. Why?	✓
(a,2)	(b,3)	(c,1)	$(b, 1)$. Why?	✗
(a,3)	(b,1)	(c,2)	$(a, 2)$. Why?	✗
(a,3)	(b,2)	(c,1)	$(a, 2)$. Why?	✗

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(a,1)	(b,2)	(c,3)	$(a, 2)$. Why?	✗
(a,1)	(b,3)	(c,2)	$(a, 2)$. Why?	✗
(a,2)	(b,1)	(c,3)	None. Why?	✓
(a,2)	(b,3)	(c,1)	$(b, 1)$. Why?	✗
(a,3)	(b,1)	(c,2)	$(a, 2)$. Why?	✗
(a,3)	(b,2)	(c,1)	$(a, 2)$. Why?	✗

- **Matching mechanism:** takes as input some report about preferences and returns as output a matching.
- What do we use as input?
  - **One-sided:** only preferences of X taken into account (e.g., house allocation)
  - **Two-sided:** preferences of both X and Y taken into account (e.g., doctors, marriage market, etc)
- In general, we want mechanisms that yield stable matchings for all the possible preference profiles.

Example (Serial dictatorship)

- One-sided mechanism
- Individuals in X are ordered by the designer: $(x_1, \ldots, x_n)$
- x_1 chooses some y from Y

  (the top Y-individual in x_1's reported ranking)
- x_2 chooses some y from those still available in Y

  (the top available Y-individual in x_2's reported ranking)

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Proposition

Serial dictatorship does not necessarily yield a stable matching.

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$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Proposition

Serial dictatorship does not necessarily yield a stable matching. Explain.

Preferences of $1 : a \succ^1 b \succ^1 c$	Preferences of $a : 2 \succ^a 1 \succ^a 3$
Preferences of $2 : a \succ^2 c \succ^2 b$	Preferences of $b : 1 \succ^b 3 \succ^b 2$
Preferences of $3 : b \succ^3 c \succ^3 a$	Preferences of $c : 3 \succ^c 2 \succ^c 1$

- Natural order: $(1, 2, 3)$
- Matching by serial dictatorship: $(a, 1), (c, 2), (b, 3)$
- Not stable: $(a, 2)$ is a blocking pair

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 c \succ^2 b$ Preferences of b : $1 \succ^b 3 \succ^b 2$

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Preferences of $3 : b \succ^3 c \succ^3 a$ Preferences of $c : 3 \succ^c 2 \succ^c 1$

- Natural order: $(1, 2, 3)$
- Matching by serial dictatorship: $(a, 1), (c, 2), (b, 3)$
- Not stable: $(a, 2)$ is a blocking pair

$$\text{Preferences of } 1: a \succ^1 b \succ^1 c \quad \text{Preferences of } a: 2 \succ^a 1 \succ^a 3$$
$$\text{Preferences of } 2: a \succ^2 c \succ^2 b \quad \text{Preferences of } b: 1 \succ^b 3 \succ^b 2$$
$$\text{Preferences of } 3: b \succ^3 c \succ^3 a \quad \text{Preferences of } c: 3 \succ^c 2 \succ^c 1$$

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- Natural order: $(1, 2, 3)$
- Matching by serial dictatorship: $(a, 1), (c, 2), (b, 3)$
- Not stable: $(a, 2)$ is a blocking pair

Example (Minimizing aggregate rank)

- We give a score to each pair (x, y) :

$$I(x, y) = n_x(y) + n_y(x),$$

  where $n_x(y)$ is the place of y in x's reported ranking
- It reflects how disliked this match is (by x and y themselves)
- Then for each matching μ we compute the sum

$$\sum_{x \in X} I(x, \mu(x))$$

- The mechanism chooses a matching with the lowest sum

Report of 1 : $a \succ^1 b \succ^1 c$

Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$

Matchings			Scores	Total Score
(a,1)	(b,2)	(c,3)	(1+1)+(3+2)+(1+2)	10

Report of 1 : $a \succ^1 b \succ^1 c$

Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

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Matchings			Scores	Total Score
(a,1)	(b,2)	(c,3)	(1+1)+(3+2)+(1+2)	10
(a,1)	(b,3)	(c,2)	(1+1)+(3+3)+(2+3)	13
(a,2)	(b,1)	(c,3)	(1+2)+(2+1)+(1+2)	9
(a,2)	(b,3)	(c,1)	(1+2)+(3+3)+(3+1)	13
(a,3)	(b,1)	(c,2)	(2+3)+(2+1)+(2+3)	13
(a,3)	(b,2)	(c,1)	(2+3)+(3+2)+(3+1)	14

Report of 1 : $a \succ^1 b \succ^1 c$

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Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$

Matchings			Scores	Total Score
(a,1)	(b,2)	(c,3)	(1+1)+(3+2)+(1+2)	10
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(a,2)	(b,1)	(c,3)	(1+2)+(2+1)+(1+2)	9
(a,2)	(b,3)	(c,1)	(1+2)+(3+3)+(3+1)	13
(a,3)	(b,1)	(c,2)	(2+3)+(2+1)+(2+3)	13
(a,3)	(b,2)	(c,1)	(2+3)+(3+2)+(3+1)	14

Proposition

Minimizing aggregate rank does not always yield stable matchings.

Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$

Matchings			Total Score	
(a,1)	(b,2)	(c,3)	10	
(a,1)	(b,3)	(c,2)	13	
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Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$

Matchings			Total Score	
(a,1)	(b,2)	(c,3)	10	
(a,1)	(b,3)	(c,2)	13	
(a,2)	(b,1)	(c,3)	9	$(a, 1)$ is a blocking pair
(a,2)	(b,3)	(c,1)	13	
(a,3)	(b,1)	(c,2)	13	
(a,3)	(b,2)	(c,1)	14	

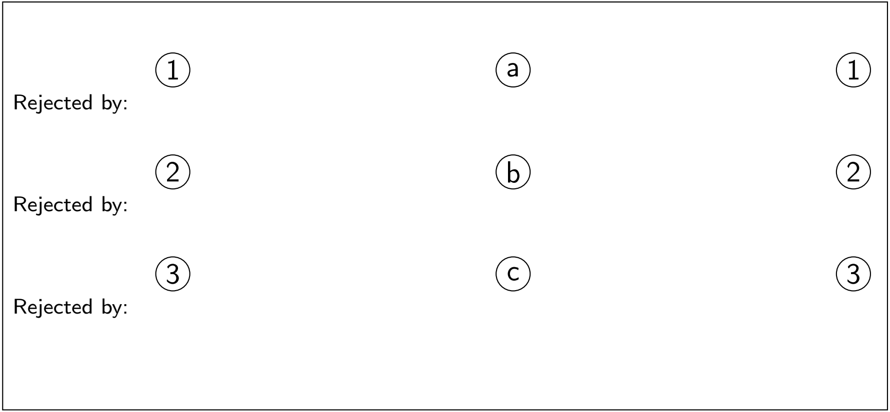
Example (Gale-Shapley / deferred acceptance algorithm)

- **Round 1:** Each x proposes a match with some y. Each y selects one of the x's that have proposed a match with her. We say that y **rejects** x, if x made a proposal to y but y selected someone else. This results to some **tentative matches**.
- **Round 2:** Each x's that is not tentatively matched in Round 1 makes a new proposal to some y that has not rejected him. Each y selects from the x tentatively assigned to her in Round 1 and the new x's that proposed to her in Round 2. This results to some new **tentative matches**.
- **Round 3:** Each x's that is not tentatively matched in Round 2 makes a new proposal to some y that has not rejected him (in Round 1 or Round 2). Each y selects from the x tentatively assigned to her in Round 2 and the new x's that proposed to her in Round 3. This results to some new **tentative matches**.
- We continue until the tentative matching involves everyone.

Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

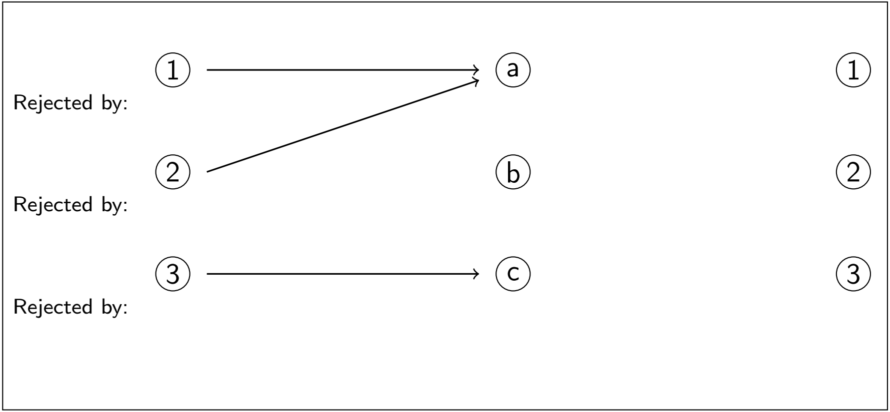
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

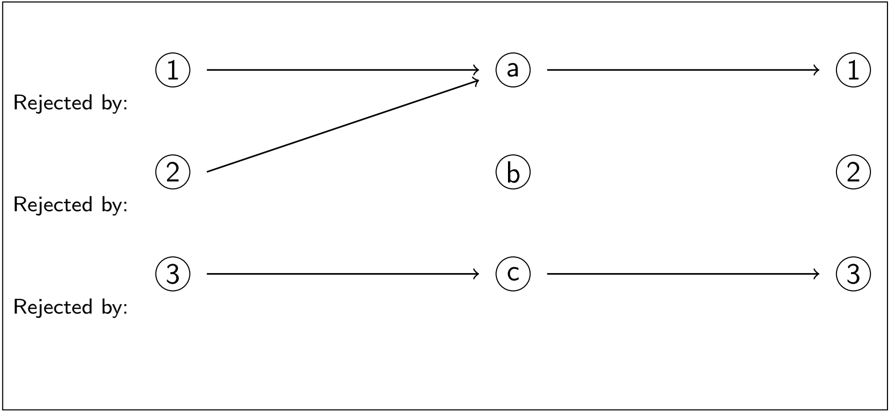
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

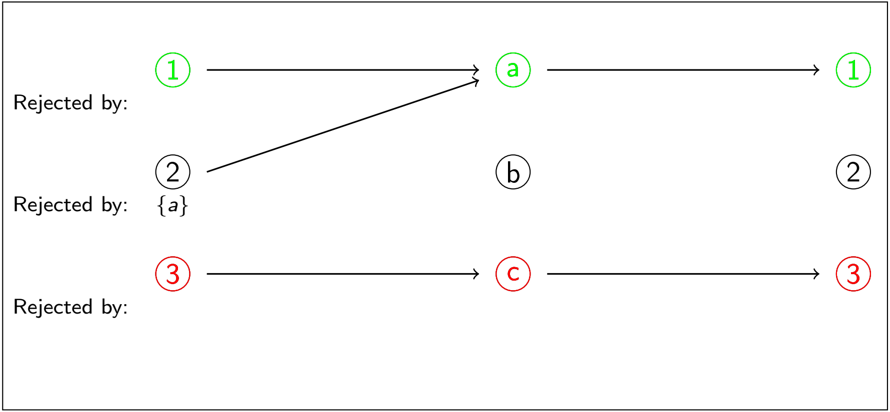
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

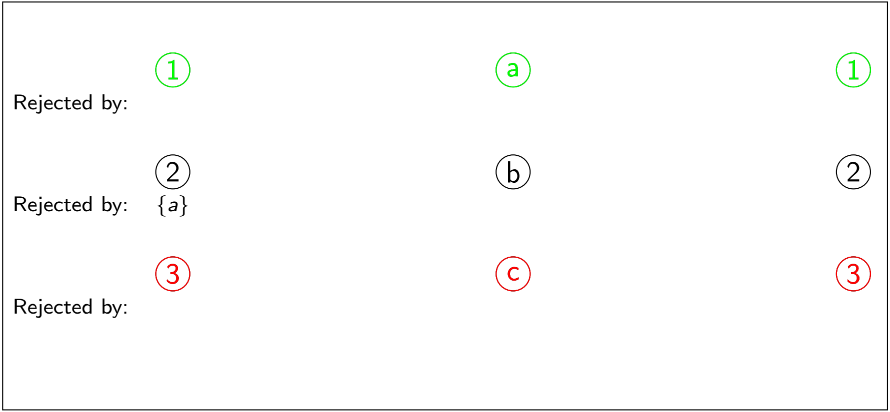
Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

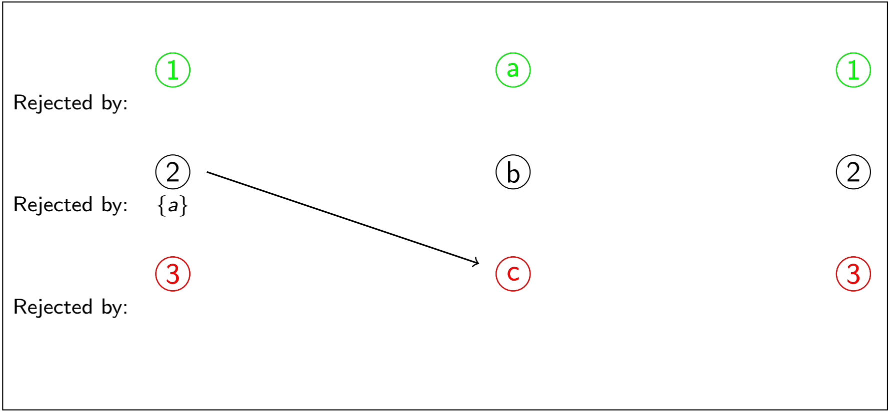
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

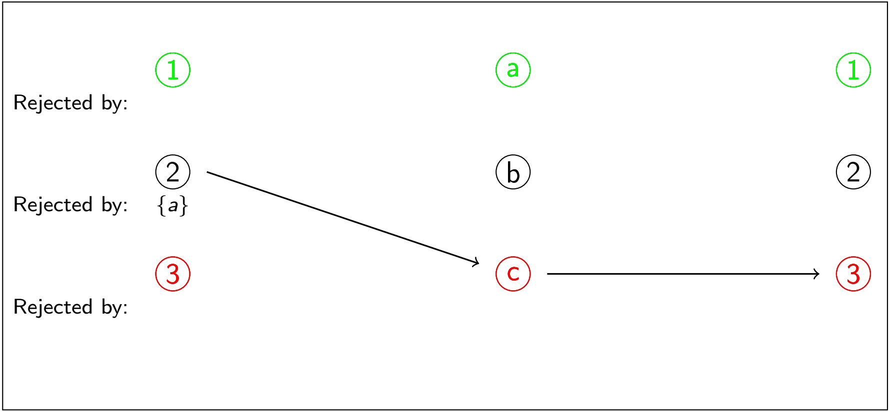
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

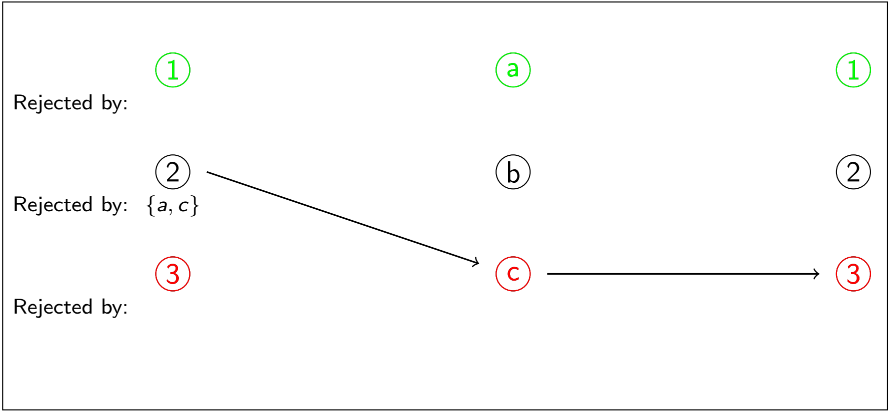
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

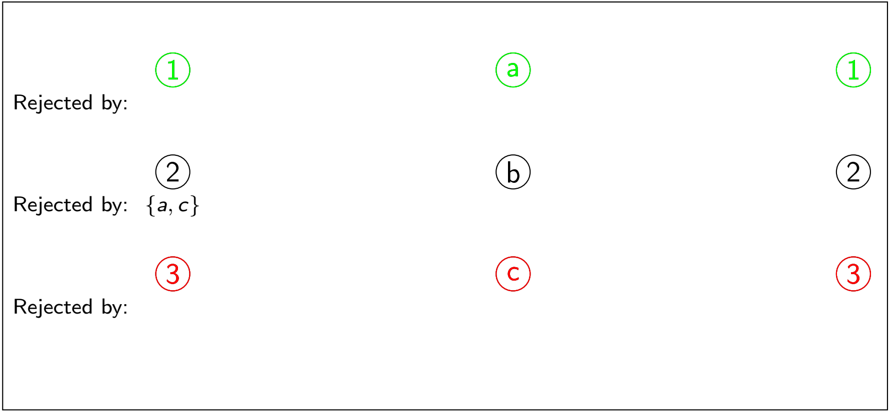
Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$

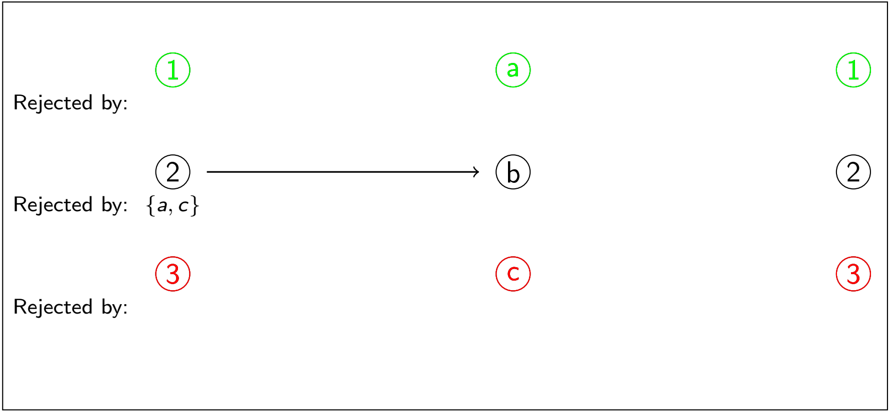
Report of 2 : $a \succ^2 c \succ^2 b$

Report of 3 : $c \succ^3 a \succ^3 b$

Report of a : $1 \succ^a 2 \succ^a 3$

Report of b : $1 \succ^b 2 \succ^b 3$

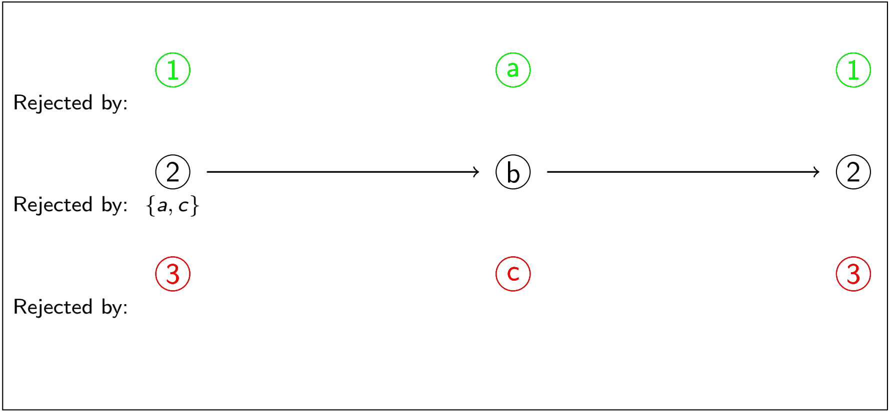
Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

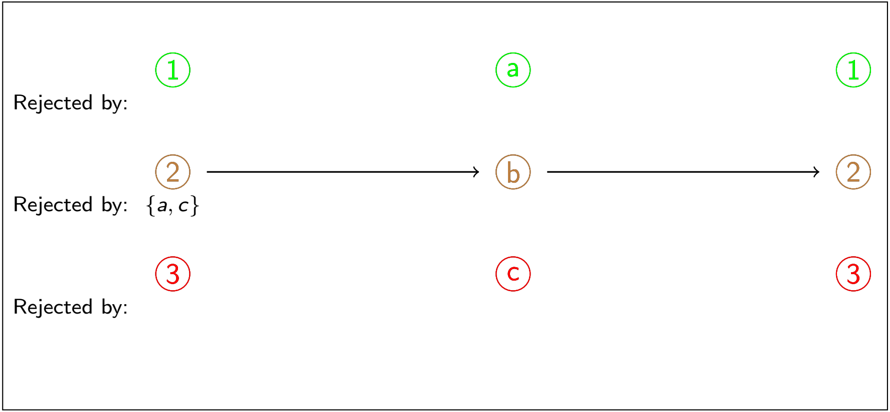
Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

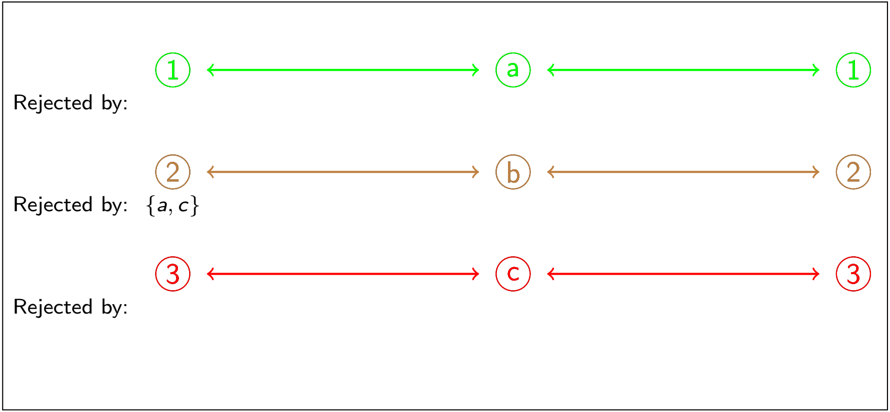
Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

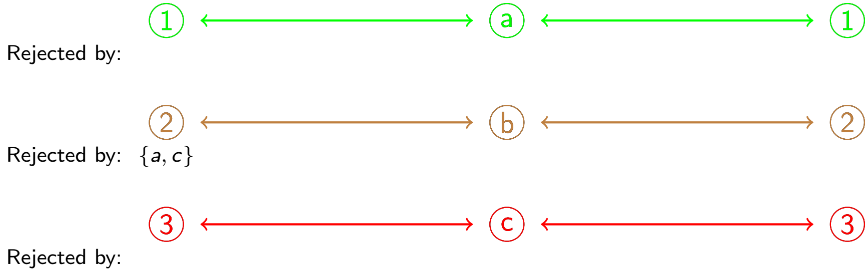
Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$

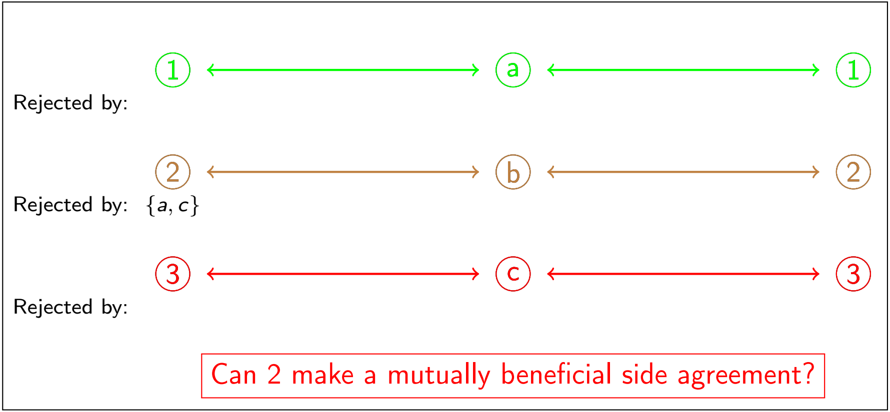


Can 1 make a mutually beneficial side agreement?

Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

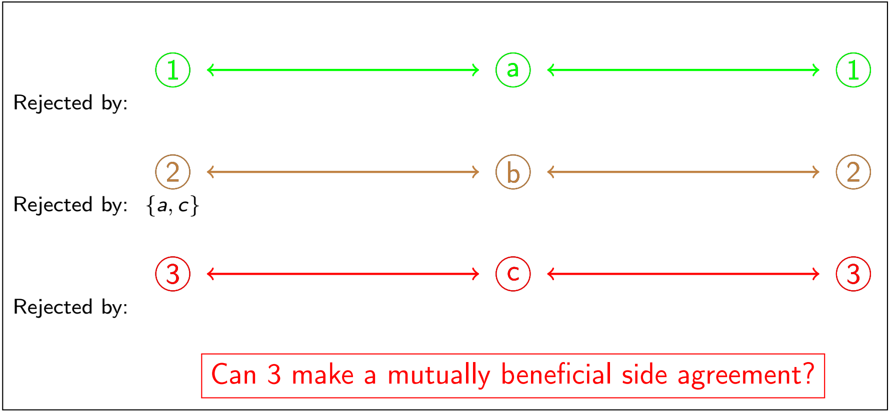
Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Report of 1 : $a \succ^1 b \succ^1 c$ Report of a : $1 \succ^a 2 \succ^a 3$

Report of 2 : $a \succ^2 c \succ^2 b$ Report of b : $1 \succ^b 2 \succ^b 3$

Report of 3 : $c \succ^3 a \succ^3 b$ Report of c : $1 \succ^c 3 \succ^c 2$



Theorem

For any preference profile, the Gale-Shapley algorithm always yields a stable matching.

Proposition

Take the stable matching that we obtain with the Gale-Shapley algorithm, when individuals is X start proposing:

1. *This is the best stable matching for any individual in X*
2. *This is the worst stable matching for any individual in Y.*

Preferences of 1 : $a \succ^1 b \succ^1 c$

Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$

Stable Matchings			**For those in $X = \{1,2,3\}$**	**For those in $Y = \{a,b,c\}$**
(a,2)	(b,1)	(c,3)	best stable matching	worst stable matching
(a,2)	(b,3)	(c,1)	worst stable matching	best stable matching

Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 b \succ^2 c$ Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of 3 : $c \succ^3 a \succ^3 b$ Preferences of c : $1 \succ^c 3 \succ^c 2$

(1) (a) (1)

Rejected by:

(2) (b) (2)

Rejected by:

(3) (c) (3)

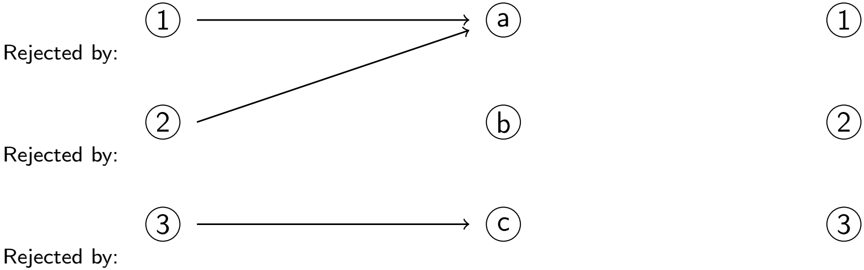
Rejected by:

Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 b \succ^2 c$ Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of 3 : $c \succ^3 a \succ^3 b$ Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

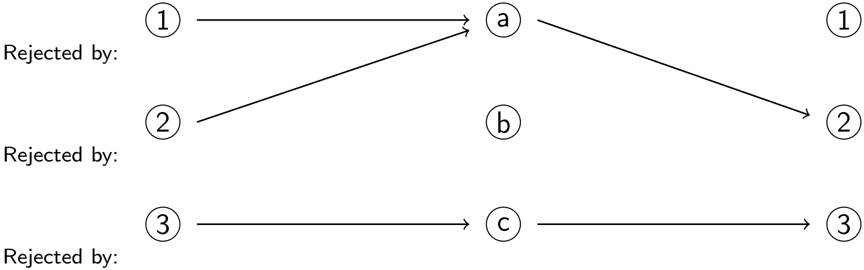
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1,2,3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

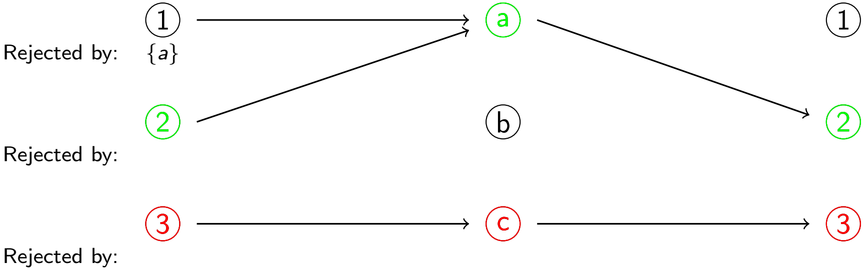
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$ | Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 b \succ^2 c$ | Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of 3 : $c \succ^3 a \succ^3 b$ | Preferences of c : $1 \succ^c 3 \succ^c 2$

	1	a	1
Rejected by:	$\{a\}$		
	2	b	2
Rejected by:			
	3	c	3
Rejected by:			

Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

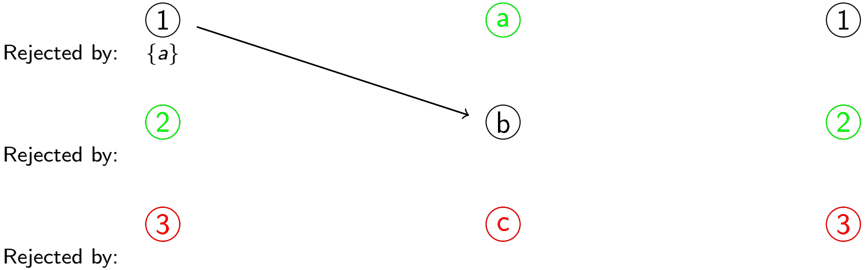
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

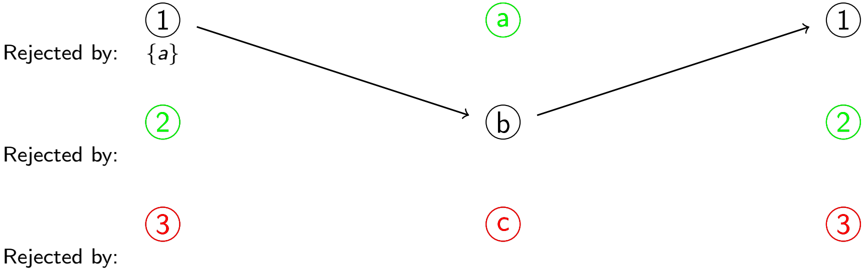
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

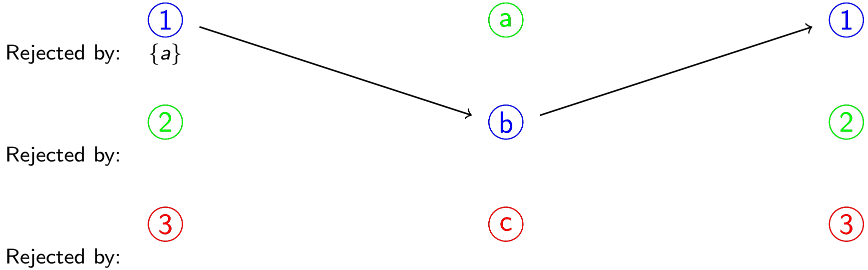
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

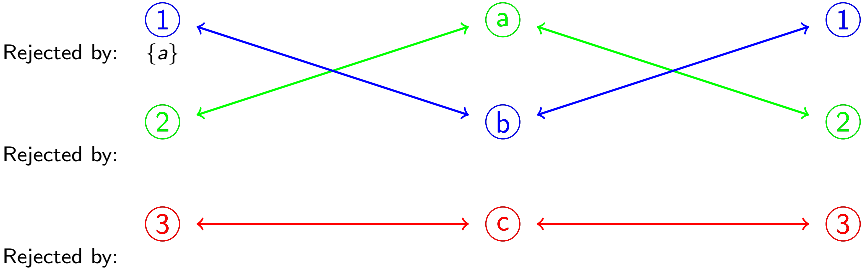
Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$

	1	a	1
Rejected by:	$\{a\}$		
	2	b	2
Rejected by:			
	3	c	3
Rejected by:			

Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$	Preferences of a : $2 \succ^a 1 \succ^a 3$
Preferences of 2 : $a \succ^2 b \succ^2 c$	Preferences of b : $3 \succ^b 2 \succ^b 1$
Preferences of 3 : $c \succ^3 a \succ^3 b$	Preferences of c : $1 \succ^c 3 \succ^c 2$

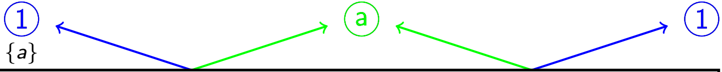


Individuals in $X = \{1, 2, 3\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 b \succ^2 c$ Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of 3 : $c \succ^3 a \succ^3 b$ Preferences of c : $1 \succ^c 3 \succ^c 2$



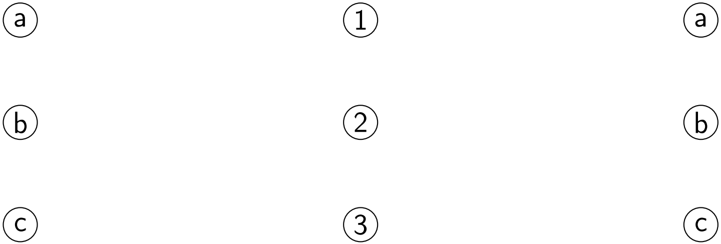
Rejected by: $\{a\}$

Stable Matchings			For those in $X = \{1, 2, 3\}$	For those in $Y = \{a, b, c\}$
(a,2)	**(b,1)**	**(c,3)**	best stable matching	worst stable matching
(a,2)	(b,3)	(c,1)	worst stable matching	best stable matching

Rejected by:

Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of $1 : a \succ^1 b \succ^1 c$	Preferences of $a : 2 \succ^a 1 \succ^a 3$
Preferences of $2 : a \succ^2 b \succ^2 c$	Preferences of $b : 3 \succ^b 2 \succ^b 1$
Preferences of $3 : c \succ^3 a \succ^3 b$	Preferences of $c : 1 \succ^c 3 \succ^c 2$



Rejected by:

Rejected by:

Rejected by:

Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

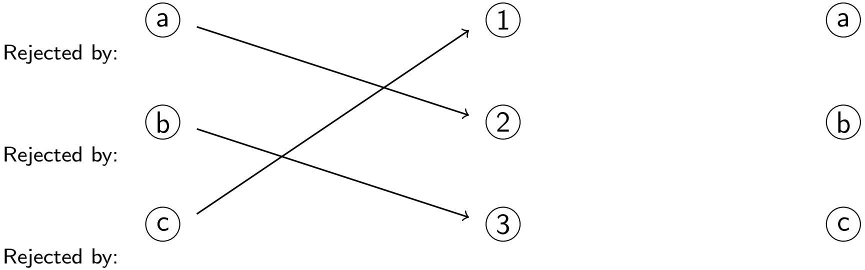
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

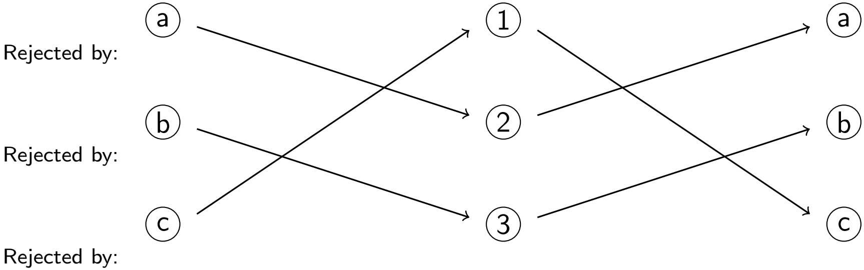
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

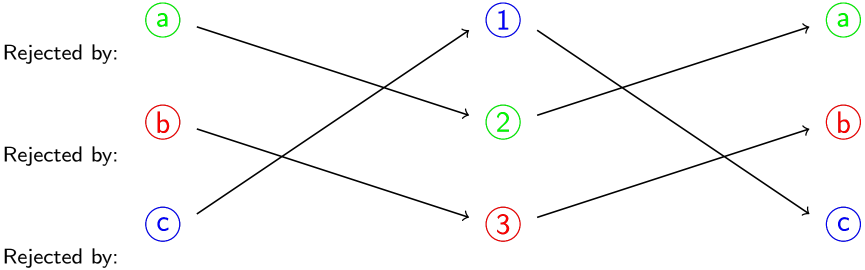
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$

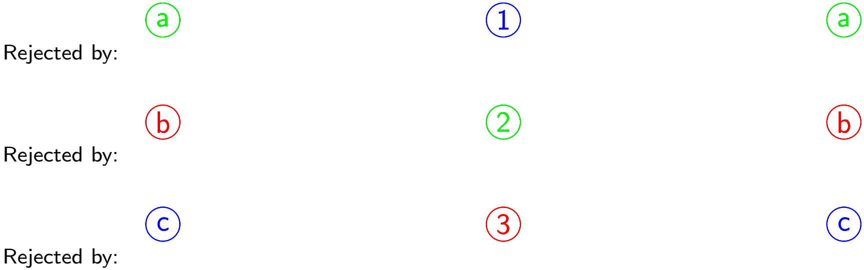


Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$ Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of 2 : $a \succ^2 b \succ^2 c$ Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of 3 : $c \succ^3 a \succ^3 b$ Preferences of c : $1 \succ^c 3 \succ^c 2$



Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

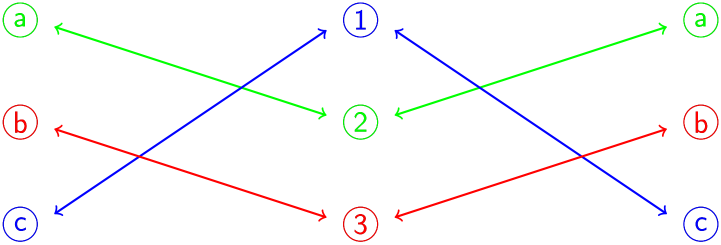
Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$

Rejected by:

Rejected by:

Rejected by:



Individuals in $Y = \{a, b, c\}$ start proposing

Preferences of 1 : $a \succ^1 b \succ^1 c$

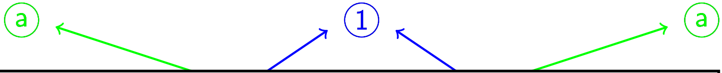
Preferences of 2 : $a \succ^2 b \succ^2 c$

Preferences of 3 : $c \succ^3 a \succ^3 b$

Preferences of a : $2 \succ^a 1 \succ^a 3$

Preferences of b : $3 \succ^b 2 \succ^b 1$

Preferences of c : $1 \succ^c 3 \succ^c 2$



Rejected by:

Stable Matchings			For those in $X = \{1, 2, 3\}$	For those in $Y = \{a, b, c\}$
(a,2)	(b,1)	(c,3)	best stable matching	worst stable matching
(a,2)	**(b,3)**	**(c,1)**	worst stable matching	best stable matching

Rejected by:

- At home you need to read Chapters 17-18 of the textbook Edition 2020).
- Same instructions as last week apply